

# $W_{1+\infty}$ $n$ -algebra

Chun-Hong Zhang<sup>a</sup>, Lu Ding<sup>b</sup>, Zhao-Wen Yan<sup>c</sup>,

Ke Wu<sup>a,d</sup> and Wei-Zhong Zhao<sup>a,d, 1</sup>

<sup>a</sup>*School of Mathematical Sciences, Capital Normal University, Beijing 100048, China*

<sup>b</sup>*Institute of Applied Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China*

<sup>c</sup>*School of Mathematical Sciences, Inner Mongolia University, Hohhot 010021, China*

<sup>d</sup>*Beijing Center for Mathematics and Information Interdisciplinary Sciences, Beijing 100048, China*

## Abstract

We present the  $W_{1+\infty}$   $n$ -algebra which satisfies the generalized Jacobi and Bremner identities for the  $n$  even and odd cases, respectively. We note that the  $W_{1+\infty}$   $2n$ -algebra is the so-called generalized Lie algebra. Its remarkable property is that there exists the nontrivial sub- $2n$ -algebra in which the structure constants are determined by the Vandermonde determinant. The central extension terms of the  $W_{1+\infty}$  sub- $2n$ -algebra are also presented. We investigate the  $W_{1+\infty}$   $n$ -algebra in the Landau problem. Furthermore we discuss the case of the many-body system in the lowest Landau level.

KEYWORDS: Conformal and W Symmetry,  $n$ -algebra, Landau problem

---

<sup>1</sup>Corresponding author: zhaowz@cnu.edu.cn

# 1 Introduction

Recently there has been a renewal of interest in  $n$ -ary algebras. One found that there are the important applications of the  $n$ -ary algebras in the string theory [1, 2] and condensed matter physics [3]-[6]. The infinite-dimensional algebras have been vigorously investigated in the literature. A flurry of recent work has been focused on the study of the infinite-dimensional 3-algebras, such as Virasoro-Witt 3-algebra [7, 8], (super)  $w_\infty$  3-algebra [9, 10] and  $SDiff(T^3)$  3-algebra [11]. The  $W_\infty$  algebra is the higher-spin extensions of the Virasoro algebra [12, 13]. It naturally arise in various physical systems. Chakraborty *et al.* [9] investigated the  $W_\infty$  3-algebra and found that it does not satisfy the Filippov condition or fundamental identity (FI) condition. By applying a double scaling limits on the generators, they obtained the classical  $w_\infty$  3-algebra which satisfies the FI.

The Landau problem is an interesting physical problem which deals with the motion of a charged particle in a plane orthogonal to a uniform magnetic field. It was investigated for the first time by Landau [14]. It was found that the spectrum consists of infinitely degenerate energy levels, i.e., the so-called Landau levels. As the curved generalization of the Landau problem, Haldane [15] considered a charged particle moving on a two-dimensional sphere  $S^2 \sim SU(2)/U(1)$  in the field of magnetic monopole placed at the center. Furthermore one has also constructed the superextensions of the Landau problem [16]-[23]. The Landau problem and its generalizations have received much attention due to their important applications in various areas. As the most important application, the Landau problem is the cornerstone of the quantum Hall effect (QHE) [24, 25]. The QHE is considered to arise from condensation of planar electrons in a magnetic field into states of incompressible quantum fluid. Two-dimensional electrons in an external uniform magnetic field occupy highly degenerate Landau levels. At low temperatures and large fields, one found that the electrons have strong quantum correlations which lead to collective motion and macroscopical quantum effects.

The infinite-dimensional symmetry for the Landau problem has been well investigated. One found that there exists  $W_{1+\infty}$  algebra in the free electron theory of Landau levels [26, 27]. The  $W_{1+\infty}$  algebra also appears in the QHE [26]-[31]. It plays an important role for the incompressibility which is a property of the bulk of the electron liquid. The  $A$ -class topological insulators can be regarded as a higher dimensional counterpart of the quantum Hall effect. Estienne *et al.* [3] investigated the  $D$ -algebra structure of topological insulators and found that there are the close relations between quantum Nambu bracket in even dimensions and  $A$ -class topological insulator.

In this paper, we shall present the  $W_{1+\infty}$   $n$ -algebra and explore its remarkable properties. Then based on the well-known structure of the  $W_{1+\infty}$  algebra in the Landau problem, we shall

show that this physical system has the  $W_{1+\infty}$   $n$ -algebraic structure. Moreover we shall discuss the case of the many-body problem in the lowest Landau level.

## 2 $W_{1+\infty}$ $n$ -algebra

Let us take the operators

$$W_m^r = z^{m+r-1} \left( \frac{\partial}{\partial z} \right)^{r-1}, \quad r \in \mathbb{Z}_+, m \in \mathbb{N}. \quad (1)$$

We then obtain the  $W_{1+\infty}$  algebra [26]

$$[W_{m_1}^{r_1}, W_{m_2}^{r_2}] = \left( \sum_{\alpha=0}^{r_1-1} C_{r_1-1}^\alpha A_{m_2+r_2-1}^\alpha - \sum_{\alpha=0}^{r_2-1} C_{r_2-1}^\alpha A_{m_1+r_1-1}^\alpha \right) W_{m_1+m_2}^{r_1+r_2-1-\alpha}, \quad (2)$$

where  $A_n^\alpha = \begin{cases} n(n-1)\cdots(n-\alpha+1), & \alpha \leq n \\ 0, & \alpha > n, \end{cases}$  and  $C_n^\alpha = \frac{n(n-1)\cdots(n-\alpha+1)}{\alpha!}$ .

Taking  $r_1 = r_2 = 2$  in (2), it gives the Virasoro-Witt algebra

$$\begin{aligned} [W_{m_1}^2, W_{m_2}^2] &= \begin{vmatrix} 1 & 1 \\ m_1 & m_2 \end{vmatrix} W_{m_1+m_2}^2 \\ &= (m_2 - m_1) W_{m_1+m_2}^2. \end{aligned} \quad (3)$$

The  $n$ -bracket is defined by [8, 32]

$$\begin{aligned} [B_{i_1}, B_{i_2}, \dots, B_{i_n}] &= \sum_{\sigma \in S_n} (-1)^{\pi(\sigma)} B_{i_{\sigma(1)}} \cdots B_{i_{\sigma(n)}} \\ &= \sum_{l=1}^n (-1)^{1+l} B_{i_l} [B_{i_1}, \dots, \hat{B}_{i_l}, \dots, B_{i_n}], \end{aligned} \quad (4)$$

where  $\pi(\sigma) = 0, 1$  is the even or odd parity of the permutation  $\sigma$  in the group  $S_n$  of permutations of the indices  $(1, \dots, n)$ , without the weight one factor  $1/n!$ , the hat symbol  $\hat{B}_{i_l}$  stands for the term  $B_{i_l}$  that is omitted.

In terms of the Lévi-Civita symbol, i.e.,

$$\epsilon_{j_1 \dots j_p}^{i_1 \dots i_p} = \det \begin{pmatrix} \delta_{j_1}^{i_1} & \cdots & \delta_{j_p}^{i_1} \\ \vdots & & \vdots \\ \delta_{j_1}^{i_p} & \cdots & \delta_{j_p}^{i_p} \end{pmatrix}, \quad (5)$$

the  $n$ -bracket can be expressed as

$$[B_{i_1}, B_{i_2}, \dots, B_{i_n}] = \epsilon_{12 \dots n}^{\sigma_1 \sigma_2 \dots \sigma_n} B_{i_{\sigma_1}} B_{i_{\sigma_2}} \cdots B_{i_{\sigma_n}}. \quad (6)$$

Substituting the operators (1) into (6), we obtain the  $W_{1+\infty}$   $n$ -algebra

$$\begin{aligned}
[W_{m_1}^{r_1}, W_{m_2}^{r_2}, \dots, W_{m_n}^{r_n}] &= \epsilon_{12\dots n}^{i_1 i_2 \dots i_n} W_{m_{i_1}}^{r_{i_1}} W_{m_{i_2}}^{r_{i_2}} \dots W_{m_{i_n}}^{r_{i_n}} \\
&= \epsilon_{12\dots n}^{i_1 i_2 \dots i_n} \sum_{\alpha_1=0}^{\beta_1} \sum_{\alpha_2=0}^{\beta_2} \dots \sum_{\alpha_{n-1}=0}^{\beta_{n-1}} C_{\beta_1}^{\alpha_1} C_{\beta_2}^{\alpha_2} \dots C_{\beta_{n-1}}^{\alpha_{n-1}} \cdot A_{m_{i_2}+r_{i_2}-1}^{\alpha_1} \\
&\quad A_{m_{i_3}+r_{i_3}-1}^{\alpha_2} \dots A_{m_{i_n}+r_{i_n}-1}^{\alpha_{n-1}} W_{m_1+m_2+\dots+m_n}^{r_1+\dots+r_n-(n-1)-\alpha_1-\dots-\alpha_{n-1}}, \quad (7)
\end{aligned}$$

where

$$\beta_k = \begin{cases} r_{i_1} - 1, & k = 1, \\ \sum_{j=1}^k r_{i_j} - k - \sum_{i=1}^{k-1} \alpha_i, & 2 \leq k \leq n-1. \end{cases} \quad (8)$$

Let us first focus on the  $n$  even case. It is known that when  $n$  is even, the  $n$ -bracket (6) with arbitrary associative operators satisfies the generalized Jacobi identity (GJI) [32]

$$\epsilon_{12\dots 2n-1}^{i_1 i_2 \dots i_{2n-1}} [[B_{i_1}, B_{i_2} \dots, B_{i_n}], B_{i_{n+1}}, \dots, B_{i_{2n-1}}] = 0. \quad (9)$$

Due to the associative operators  $W_m^r$  (1), we can confirm that the  $W_{1+\infty}$   $n$ -algebra with  $n$  even (7) is a generalized Lie algebra or higher order Lie algebra which satisfies the GJI (9).

For the  $W_{1+\infty}$  algebra, it is known that there is only a nontrivial subalgebra, i.e., the Virasoro-Witt algebra (3). Since the  $W_{1+\infty}$   $2n$ -algebra is a generalized Lie algebra, we therefore proceed by seeking its subalgebras. Let us consider the generators  $W_m^{n+1}$  with any fixed superindex  $n+1 \geq 2$ . Performing straightforward calculations, we find that they form a sub- $2n$ -algebra

$$\begin{aligned}
[W_{m_1}^{n+1}, W_{m_2}^{n+1}, \dots, W_{m_{2n}}^{n+1}] &= \epsilon_{12\dots 2n}^{i_1 i_2 \dots i_{2n}} \sum_{\alpha_1=0}^{\beta_1} \sum_{\alpha_2=0}^{\beta_2} \dots \sum_{\alpha_{2n-1}=0}^{\beta_{2n-1}} \\
&\quad C_{\beta_1}^{\alpha_1} C_{\beta_2}^{\alpha_2} \dots C_{\beta_{2n-1}}^{\alpha_{2n-1}} A_{m_{i_2}+n}^{\alpha_1} A_{m_{i_3}+n}^{\alpha_2} \dots A_{m_{i_{2n}}+n}^{\alpha_{2n-1}} W_{m_{i_1}+m_{i_2}+\dots+m_{i_{2n}}}^{2n^2+1-\alpha_1-\dots-\alpha_{2n-1}} \\
&= \sum_{(\alpha_1, \alpha_2, \dots, \alpha_{2n-1}) \in S_{2n-1}} C_{\beta_1}^{\alpha_1} C_{\beta_2}^{\alpha_2} \dots C_{\beta_{2n-1}}^{\alpha_{2n-1}} \epsilon_{12\dots (2n-1)}^{\alpha_1 \alpha_2 \dots \alpha_{2n-1}} V_{2n} W_{m_1+m_2+\dots+m_{2n}}^{n+1}, \quad (10)
\end{aligned}$$

where we denote  $\beta_1 = n, \beta_k = kn - \sum_{i=1}^{k-1} \alpha_i, (k \geq 2)$ ,  $V_{2n}$  is the so-called Vandermonde determinant

$$V_{2n} = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ m_1 & m_2 & \cdots & m_{2n-1} & m_{2n} \\ m_1^2 & m_2^2 & \cdots & m_{2n-1}^2 & m_{2n}^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_1^{2n-2} & m_2^{2n-2} & \cdots & m_{2n-1}^{2n-2} & m_{2n}^{2n-2} \\ m_1^{2n-1} & m_2^{2n-1} & \cdots & m_{2n-1}^{2n-1} & m_{2n}^{2n-1} \end{vmatrix} = \prod_{1 \leq j < k \leq 2n} (m_k - m_j) \quad (11)$$

and  $S_{2n-1}$  is the permutation group of  $\{1, 2, \dots, 2n-1\}$ .

Let us take the scaled generators  $W_m^{n+1} \rightarrow Q^{-\frac{1}{2n-1}} W_m^{n+1}$ , where the scaling coefficient  $Q$  is given by  $Q = \sum_{(\alpha_1, \alpha_2, \dots, \alpha_{2n-1}) \in S_{2n-1}} C_{\beta_1}^{\alpha_1} C_{\beta_2}^{\alpha_2} \cdots C_{\beta_{2n-1}}^{\alpha_{2n-1}} \epsilon_{12 \dots (2n-1)}^{\alpha_1 \alpha_2 \dots \alpha_{2n-1}}$ . Thus (10) can be rewritten as

$$[W_{m_1}^{n+1}, W_{m_2}^{n+1}, \dots, W_{m_{2n}}^{n+1}] = V_{2n} W_{m_1+m_2+\dots+m_{2n}}^{n+1}. \quad (12)$$

As a generalized Lie algebra, we note that the structure constants of the  $W_{1+\infty}$  sub- $2n$ -algebra (12) are determined by the Vandermonde determinant. A particular case is that when  $n = 1$ , (12) gives the Virasoro-Witt algebra (3).

According to the definition of  $n$ -bracket (4) and using (12), we obtain

$$\begin{aligned} [W_{m_1}^{n+1}, \dots, W_{m_{2n+1}}^{n+1}] &= \sum_{i=1}^{2n+1} (-1)^{1+i} W_{m_i}^{n+1} [W_{m_1}^{n+1}, \dots, \hat{W}_{m_i}^{n+1}, \dots, W_{m_{2n+1}}^{n+1}] \\ &= \sum_{\alpha=0}^n C_n^\alpha \begin{vmatrix} A_{\bar{m}-m_1}^\alpha & \cdots & A_{\bar{m}-m_i}^\alpha & \cdots & A_{\bar{m}-m_{2n+1}}^\alpha \\ 1 & \cdots & 1 & \cdots & 1 \\ m_1 & \cdots & m_i & \cdots & m_{2n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ m_1^{2n-1} & \cdots & m_i^{2n-1} & \cdots & m_{2n+1}^{2n-1} \end{vmatrix} W_{m_1+\dots+m_{2n+1}}^{2n+1-\alpha} \\ &= 0, \end{aligned} \quad (13)$$

where  $\bar{m} = m_1 + \dots + m_{2n+1} + n$ .

Due to (13), we have

$$[W_{m_1}^{n+1}, \dots, W_{m_k}^{n+1}] = 0, \quad k \geq 2n+1. \quad (14)$$

As the examples, below we list the several sub- $2n$ -algebras.

- Sub-4-algebra

$$\begin{aligned} [W_{m_1}^3, W_{m_2}^3, W_{m_3}^3, W_{m_4}^3] &= (m_4 - m_3)(m_4 - m_2)(m_4 - m_1)(m_3 - m_2) \\ &\quad (m_3 - m_1)(m_2 - m_1) W_{m_1+m_2+m_3+m_4}^3. \end{aligned} \quad (15)$$

When  $n \geq 5$ , we have the null  $n$ -algebra

$$[W_{m_1}^3, W_{m_2}^3, \dots, W_{m_n}^3] = 0. \quad (16)$$

- Sub-6-algebra

$$[W_{m_1}^4, W_{m_2}^4, \dots, W_{m_6}^4] = \prod_{1 \leq j < k \leq 6} (m_k - m_j) W_{\sum_{i=1}^6 m_i}^4. \quad (17)$$

When  $n \geq 7$ , we have the null  $n$ -algebra

$$[W_{m_1}^4, W_{m_2}^4, \dots, W_{m_n}^4] = 0. \quad (18)$$

- Sub-8-algebra

$$[W_{m_1}^5, W_{m_2}^5, \dots, W_{m_8}^5] = \prod_{1 \leq j < k \leq 8} (m_k - m_j) W_{\sum_{i=1}^8 m_i}^5. \quad (19)$$

When  $n \geq 9$ , we have the null  $n$ -algebra

$$[W_{m_1}^5, W_{m_2}^5, \dots, W_{m_n}^5] = 0. \quad (20)$$

Let us turn to the  $n$  odd case. In this case, we find that the  $W_{1+\infty}$   $n$ -algebra does not satisfy the GJI (9). For odd  $n$ -brackets built from associative operator products, it is known that they satisfy the generalized Bremner identity (GBI) [33, 34]

$$\begin{aligned} & \epsilon_{1\dots 3n-3}^{i_1\dots i_{3n-3}} [[A, B_{i_1}, \dots, B_{i_{n-1}}], [B_{i_n}, \dots, B_{i_{2n-1}}], B_{i_{2n}}, \dots, B_{i_{3n-3}}] \\ &= \epsilon_{1\dots 3n-3}^{i_1\dots i_{3n-3}} [[A, [B_{i_1}, \dots, B_{i_n}], B_{i_{n+1}}, \dots, B_{i_{2n-2}}], B_{i_{2n-1}}, \dots, B_{i_{3n-3}}]. \end{aligned} \quad (21)$$

Thus the  $W_{1+\infty}$   $n$ -algebra (7) with  $n$  odd satisfies the GBI (21).

### 3 Central extensions of the $W_{1+\infty}$ sub- $2n$ -algebra

The central extensions of the infinite-dimensional algebras have been well investigated in the literature. Pope *et al.* constructed the centrally extended  $W_{1+\infty}$  algebra [12, 35]. It plays an important role in two-dimensional conformal field theories. In the context of integrable systems, the centerless  $W_{1+\infty}$  algebra provides a Hamiltonian structure for the KP hierarchy [36, 37]. For the centrally extended  $W_{1+\infty}$  algebra, it yields yet another Poisson structure for the KP hierarchy [38]. The question that naturally arises is whether there exist the central terms in the  $W_{1+\infty}$   $n$ -algebra. In this section, we shall give a part of affirmative answer to this question.

Now let us discuss the central extensions of (12)

$$[W_{m_1}^{n+1}, W_{m_2}^{n+1}, \dots, W_{m_{2n}}^{n+1}] = V_{2n} W_{m_1+m_2+\dots+m_{2n}}^{n+1} + C(m_1, m_2, \dots, m_{2n}). \quad (22)$$

From the skewsymmetry of (22) and the GJI (9), we obtain the following relations of the central terms:

$$C(m_1, \dots, m_i, \dots, m_j, \dots, m_{2n}) = -C(m_1, \dots, m_j, \dots, m_i, \dots, m_{2n}),$$

$$\epsilon_{12 \dots 4n-1}^{i_1 i_2 \dots i_{4n-1}} V_{2n}(m_{i_1}, m_{i_2}, \dots, m_{i_{2n}}) C(m_{i_1} + \dots + m_{i_{2n}}, m_{i_{2n+1}}, \dots, m_{i_{4n-1}}) = 0. \quad (23)$$

It should be emphasized that the structure constants of the  $W_{1+\infty}$  sub- $2n$ -algebra are determined by the Vandermonde determinant. Owing to remarkable property of the structure constants, we may derive the following central terms from (23):

$$C(m_1, m_2, \dots, m_{2n}) = \frac{c}{12 \times 2^n n!} \sum_{(i_1 i_2 \dots i_{2n}) \in S_{2n}} \epsilon_{12 \dots 2n}^{i_1 i_2 \dots i_{2n}} \cdot \prod_{k=1}^n [(m_{i_{2k-1}}^3 - m_{i_{2k-1}}) \delta_{m_{i_{2k-1}} + m_{i_{2k}}, 0}], \quad (24)$$

where  $c$  is an arbitrary constant.

Therefore we conclude that the centrally extended  $W_{1+\infty}$  sub- $2n$ -algebra is

$$[W_{m_1}^{n+1}, W_{m_2}^{n+1}, \dots, W_{m_{2n}}^{n+1}] = V_{2n} W_{m_1 + m_2 + \dots + m_{2n}}^{n+1} + \frac{c}{12 \times 2^n n!} \sum_{(i_1 i_2 \dots i_{2n}) \in S_{2n}} \epsilon_{12 \dots 2n}^{i_1 i_2 \dots i_{2n}} \prod_{k=1}^n [(m_{i_{2k-1}}^3 - m_{i_{2k-1}}) \delta_{m_{i_{2k-1}} + m_{i_{2k}}, 0}]. \quad (25)$$

For the generators  $W_m^2$ , we immediately recognize that (25) gives the well-known Virasoro algebra

$$[W_{m_1}^2, W_{m_2}^2] = (m_2 - m_1) W_{m_1 + m_2}^2 + \frac{c}{12} (m_1^3 - m_1) \delta_{m_1 + m_2, 0}. \quad (26)$$

Let us list the next few centrally extended sub- $2n$ -algebra as follows:

- centrally extended sub-4-algebra

$$\begin{aligned} [W_{m_1}^3, W_{m_2}^3, W_{m_3}^3, W_{m_4}^3] &= (m_4 - m_3)(m_4 - m_2)(m_4 - m_1)(m_3 - m_2) \\ &\quad (m_3 - m_1)(m_2 - m_1) W_{m_1 + m_2 + m_3 + m_4}^3 \\ &\quad + \frac{c}{12} [(m_1^3 - m_1)(m_3^3 - m_3) \delta_{m_1 + m_2, 0} \delta_{m_3 + m_4, 0} \\ &\quad + (m_2^3 - m_2)(m_3^3 - m_3) \delta_{m_1 + m_3, 0} \delta_{m_2 + m_4, 0} \\ &\quad + (m_1^3 - m_1)(m_2^3 - m_2) \delta_{m_1 + m_4, 0} \delta_{m_2 + m_3, 0}]. \end{aligned} \quad (27)$$

- centrally extended sub-6-algebra

$$\begin{aligned} [W_{m_1}^4, W_{m_2}^4, \dots, W_{m_6}^4] &= \prod_{1 \leq j < k \leq 6} (m_k - m_j) W_{\sum_{i=1}^6 m_i}^4 \\ &\quad + \frac{c}{12 \times 2^3 3!} \sum_{(i_1 i_2 \dots i_6) \in S_6} \epsilon_{12 \dots 6}^{i_1 i_2 \dots i_6} \prod_{k=1}^3 [(m_{i_{2k-1}}^3 - m_{i_{2k-1}}) \delta_{m_{i_{2k-1}} + m_{i_{2k}}, 0}]. \end{aligned} \quad (28)$$

- centrally extended sub-8-algebra

$$\begin{aligned}
[W_{m_1}^5, W_{m_2}^5, \dots, W_{m_8}^5] &= \prod_{1 \leq j < k \leq 8} (m_k - m_j) W_{\sum_{i=1}^8 m_i}^5 \\
&+ \frac{c}{12 \times 2^4 4!} \sum_{(i_1 i_2 \dots i_8) \in S_8} \epsilon_{12 \dots 8}^{i_1 i_2 \dots i_8} \prod_{k=1}^4 [(m_{i_{2k-1}}^3 - m_{i_{2k-1}}) \delta_{m_{i_{2k-1}} + m_{i_{2k}}, 0}].
\end{aligned} \tag{29}$$

## 4 $W_{1+\infty}$ $n$ -algebra in the Landau problem

Let us consider an electron of charge  $e$  and mass  $m$  moving on a plane in presence of a constant perpendicular magnetic field  $B$ . The Hamiltonian is given by

$$H = \frac{1}{2m} (\mathbf{p} - \frac{e}{c} \mathbf{A})^2. \tag{30}$$

The Schrödinger equation for such an electron is given by

$$H\psi = \frac{1}{2m} (\mathbf{p} - \frac{e}{c} \mathbf{A})^2 \psi = E\psi, \tag{31}$$

where the momentum  $\mathbf{p} = -i\hbar\nabla$  and the gauge potential  $\mathbf{A}$  exist in the plane. Let us choose the symmetric  $\mathbf{A} = \frac{B}{2}(-y, x)$  and introduce complex variables:  $z = x + iy, \bar{z} = x - iy$ . In the following the units  $c = m = 1$  will be employed. For convenience we take  $eB = 2$ .

We may construct the harmonic oscillator operators  $a$  and  $a^\dagger$ ,

$$a = \frac{z}{2} + \hbar\bar{\partial}, \quad a^\dagger = \frac{\bar{z}}{2} - \hbar\partial, \tag{32}$$

obeying

$$[a, a^\dagger] = \hbar. \tag{33}$$

In terms of the harmonic oscillator operators  $a$  and  $a^\dagger$ , the Hamiltonian (30) can be rewritten as

$$H = aa^\dagger + a^\dagger a. \tag{34}$$

We may introduce another pair  $b$  and  $b^\dagger$  commuting with  $a$  and  $a^\dagger$ ,

$$b = \frac{\bar{z}}{2} + \hbar\partial, \quad b^\dagger = \frac{z}{2} - \hbar\bar{\partial}, \tag{35}$$

with commutation relation

$$[b, b^\dagger] = \hbar. \tag{36}$$



The angular momentum operator can be written as

$$J = b^\dagger b - a^\dagger a. \quad (37)$$

It is commuting with the Hamiltonian (34), i.e.,

$$[J, H] = 0. \quad (38)$$

The vacuum is determined by the condition  $a\psi_{0,0} = b\psi_{0,0} = 0$  and given as

$$\psi_{0,0} = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{1}{2\hbar}|z|^2\right). \quad (39)$$

The general wave function of energy  $n$  and angular momentum  $l$  is given by

$$\psi_{n,l} = \frac{(b^\dagger)^{l+n}}{\sqrt{(l+n)!}} \frac{(a^\dagger)^n}{\sqrt{n!}} \psi_{0,0} = \sqrt{\frac{n!}{(l+n)!}} (-\hbar)^n z^l L_n^l\left(\frac{z\bar{z}}{\hbar}\right) \psi_{0,0}. \quad (40)$$

where  $L_n^l(x)$  are the generalized Laguerre polynomials and  $n \geq 0, l+n \geq 0$ .

Let us take the operators [26]

$$\tilde{W}_m^r = (b^\dagger)^{m+r-1} (b)^{r-1}, \quad r \geq 1, \quad m+r \geq 1, \quad (41)$$

which are commuting with the Hamiltonian (34),

$$[\tilde{W}_m^r, H] = 0. \quad (42)$$

The action of (41) on the wave function (40) is

$$\tilde{W}_m^r \psi_{n,l} = A_{l+n}^{r-1} \sqrt{\frac{(m+n+l)!}{(l+n)!}} \psi_{n,m+l}. \quad (43)$$

For the infinite conserved operators (41), they form the  $W_{1+\infty}$  algebra [26]

$$[\tilde{W}_{m_1}^{r_1}, \tilde{W}_{m_2}^{r_2}] = \left( \sum_{\alpha=0}^{r_1-1} C_{r_1-1}^\alpha A_{m_2+r_2-1}^\alpha - \sum_{\alpha=0}^{r_2-1} C_{r_2-1}^\alpha A_{m_1+r_1-1}^\alpha \right) \hbar^\alpha \tilde{W}_{m_1+m_2}^{r_1+r_2-1-\alpha}, \quad (44)$$

and the  $W_{1+\infty}$   $n$ -algebra

$$\begin{aligned} [\tilde{W}_{m_1}^{r_1}, \tilde{W}_{m_2}^{r_2}, \dots, \tilde{W}_{m_n}^{r_n}] &= \epsilon_{12\dots n}^{i_1 i_2 \dots i_n} \sum_{\alpha_1=0}^{\beta_1} \sum_{\alpha_2=0}^{\beta_2} \dots \sum_{\alpha_{n-1}=0}^{\beta_{n-1}} C_{\beta_1}^{\alpha_1} C_{\beta_2}^{\alpha_2} \dots C_{\beta_{n-1}}^{\alpha_{n-1}} \cdot A_{m_{i_2}+r_{i_2}-1}^{\alpha_1} A_{m_{i_3}+r_{i_3}-1}^{\alpha_2} \\ &\quad \dots A_{m_{i_n}+r_{i_n}-1}^{\alpha_{n-1}} \hbar^{\alpha_1+\dots+\alpha_{n-1}} \tilde{W}_{m_1+m_2+\dots+m_n}^{r_1+\dots+r_n-(n-1)-\alpha_1-\dots-\alpha_{n-1}}. \end{aligned} \quad (45)$$

It should be pointed out that not as the generators (1), here the  $(n)$ -algebra corresponds to the so-called wedge  $W_\Lambda = \{\tilde{W}_m^r, |m| \leq r-1\}$ , plus the positive modes  $m > r-1$ . Thus the incompleteness of the algebra prevents us from considering possible central extensions.

In the classical limit, (44) reduces to the classical  $w_\infty$  algebra [12, 26]

$$\{\tilde{W}_{m_1}^{r_1}, \tilde{W}_{m_2}^{r_2}\} = -\mathbf{i}(m_2(r_1 - 1) - m_1(r_2 - 1))\tilde{W}_{m_1+m_2}^{r_1+r_2-2}, \quad (46)$$

which is the canonical symmetry of the classical case of the Hamiltonian (30).

Let us consider the  $W_{1+\infty}$  3-algebra. From (45), we may rewritten the  $W_{1+\infty}$  3-algebra as

$$\begin{aligned} [\tilde{W}_{m_1}^{r_1}, \tilde{W}_{m_2}^{r_2}, \tilde{W}_{m_3}^{r_3}] &= \hbar(r_1(m_2 - m_3) + r_2(m_3 - m_1) \\ &+ r_3(m_1 - m_2))\tilde{W}_{m_1+m_2+m_3}^{r_1+r_2+r_3-3} + O(\hbar^2). \end{aligned} \quad (47)$$

It satisfies the Bremner identity (BI) [39, 40]

$$\epsilon^{i_1 i_2 \dots i_6} [[A, [B_{i_1}, B_{i_2}, B_{i_3}], B_{i_4}], B_{i_5}, B_{i_6}] = \epsilon^{i_1 i_2 \dots i_6} [[A, B_{i_1}, B_{i_2}], [B_{i_3}, B_{i_4}, B_{i_5}], B_{i_6}], \quad (48)$$

where  $i_1, \dots, i_6$  are implicitly summed from 1 to 6.

In the classical limit  $\{, , \} = \lim_{\hbar \rightarrow 0} \frac{1}{\mathbf{i}\hbar} [, ,]$ , (47) becomes the  $w_\infty$ -3-algebra [9]

$$\{\tilde{W}_{m_1}^{r_1}, \tilde{W}_{m_2}^{r_2}, \tilde{W}_{m_3}^{r_3}\} = -\mathbf{i}(r_1(m_2 - m_3) + r_2(m_3 - m_1) + r_3(m_1 - m_2))\tilde{W}_{m_1+m_2+m_3}^{r_1+r_2+r_3-3}. \quad (49)$$

It is worth to emphasize that the  $w_\infty$ -3-algebra (49) satisfies the classical FI [41]:

$$\{A, B, \{C, D, E\}\} = \{\{A, B, C\}, D, E\} + \{C, \{A, B, D\}, E\} + \{C, D, \{A, B, E\}\}. \quad (50)$$

Thus it is a classical Nambu 3-algebra.

As the case of (12), for the the infinite conserved operators (41), they also form the sub- $2n$ -algebra

$$[\tilde{W}_{m_1}^{n+1}, \tilde{W}_{m_2}^{n+1}, \dots, \tilde{W}_{m_{2n}}^{n+1}] = \hbar^{n(2n-1)} V_{2n} \tilde{W}_{m_1+m_2+\dots+m_{2n}}^{n+1}. \quad (51)$$

For the  $2n$ -bracket (6), there is the following limiting relation [42]:

$$\frac{1}{n!} \lim_{\hbar \rightarrow 0} \left(\frac{1}{\mathbf{i}\hbar}\right)^n [B_1, B_2, \dots, B_{2n}] = \{B_1, B_2, \dots, B_{2n}\}, \quad (52)$$

where the right hand side bracket is the classical Nambu  $2n$ -bracket. In the classical limit (52), the  $W_{1+\infty}$  sub- $2n$ -algebra (51) becomes

$$\{\tilde{W}_{m_1}^{n+1}, \tilde{W}_{m_2}^{n+1}, \dots, \tilde{W}_{m_{2n}}^{n+1}\} = 0. \quad (53)$$

It implies that the quantum multibrackets represent higher order quantum effects. For the infinite conserved operators  $\tilde{W}_m^{n+1}$  with any fixed superindex, they also form the null  $k$ -algebras (14). That is to say that there does not exist any quantum effect term in these quantum multibrackets. However from (51), we note that these infinite conserved operators may form the

infinite symmetry in  $\hbar^{n(2n-1)}$ . This remarkable property represents the intrinsic symmetry in higher order of  $\hbar$ .

It is known that the angular momentum operator  $J$  (37) and the operators (41) are commuting with the Hamiltonian  $H$  (34), respectively. For the triple operators  $(\tilde{W}_m^r, J, H)$ , a direct calculation shows that

$$[\tilde{W}_m^r, J, H] \neq 0, \quad m \neq 0. \quad (54)$$

Note, however, that for the triple operators  $(\tilde{W}_0^r, J, H)$ , they are in involution with the 3-bracket structure, i.e.,

$$[\tilde{W}_0^r, J, H] = 0. \quad (55)$$

It is known that the infinite conserved operators  $\tilde{W}_m^r$  (41) commute with the Hamiltonian  $H$  (34). For the case of even  $n$ -brackets, it is easy to see

$$[\tilde{W}_{m_1}^{r_1}, \dots, \tilde{W}_{m_{n-1}}^{r_{n-1}}, H] = 0. \quad (56)$$

However (56) does not hold for the odd  $n$ -brackets.

By means of (14), we find that

$$[\tilde{W}_{m_1}^{n+1}, \dots, \tilde{W}_{m_k}^{n+1}, H] = 0, \quad k \geq 2n + 1. \quad (57)$$

Thus for the conserved operators  $\tilde{W}_m^{n+1}$  with any fixed superindex, besides they commute with the Hamiltonian  $H$  and satisfy (56), we have also the null odd multi-commutators (57) with respect to the Hamiltonian and these conserved operators.

It should be noted that the operators  $\tilde{W}_m^r$  generically involve powers of derivatives higher than one, and therefore are not generators of local coordinate transformations on the wave function. It implies that the generators  $\tilde{W}_m^r$  are quasi-local operators. The generating function of  $\tilde{W}_m^r$  is actually the finite magnetic translation.

Let us take [27]

$$T_{\mathbf{n}} = \exp\left(-\frac{B}{2}|\mathbf{n}|^2\right) \sum_{k,l=0}^{\infty} (-1)^l \frac{(n_1 + i n_2)^k}{k!} \frac{(n_1 - i n_2)^l}{l!} \tilde{W}_{k-l}^{l+1}, \quad (58)$$

where  $\mathbf{n} = (n_1, n_2)$ ,  $b = \frac{B\bar{z}}{2} + \partial$ ,  $b^\dagger = \frac{Bz}{2} - \bar{\partial}$  and the commutator between  $b$  and  $b^\dagger$  is  $[b, b^\dagger] = B$ .

For the generators  $T_{\mathbf{n}}$  (58), they form the well-known sine algebra [43, 44]

$$[T_{\mathbf{n}}, T_{\mathbf{m}}] = 2i \sin B(\mathbf{m} \times \mathbf{n}) T_{\mathbf{n}+\mathbf{m}}, \quad (59)$$

and the sine  $n$ -algebra [45]

$$\begin{aligned}
[T_{\mathbf{m}_1}, \dots, T_{\mathbf{m}_n}] &= \frac{1}{2} (\epsilon_{1\dots n}^{i_1\dots i_n} \exp(\mathbf{i}B \sum_{l < k} \mathbf{m}_{i_k} \times \mathbf{m}_{i_l}) \\
&\quad + \epsilon_{1\dots n}^{i_n\dots i_1} \exp(\mathbf{i}B \sum_{l > k} \mathbf{m}_{i_k} \times \mathbf{m}_{i_l})) T_{\mathbf{m}_1 + \dots + \mathbf{m}_n} \\
&= \frac{1}{2} \epsilon_{1\dots n}^{i_1\dots i_n} \cos(B \sum_{l < k} \mathbf{m}_{i_k} \times \mathbf{m}_{i_l}) (1 + (-1)^{\frac{n(n-1)}{2}}) T_{\mathbf{m}_1 + \dots + \mathbf{m}_n} \\
&\quad + \frac{\mathbf{i}}{2} \epsilon_{1\dots n}^{i_1\dots i_n} \sin(B \sum_{l < k} \mathbf{m}_{i_k} \times \mathbf{m}_{i_l}) (1 - (-1)^{\frac{n(n-1)}{2}}) T_{\mathbf{m}_1 + \dots + \mathbf{m}_n}.
\end{aligned} \tag{60}$$

When  $n$  is even, (60) is a generalized Lie algebra.

Corresponding to (50), the FI for the quantal ternary algebras is

$$[A, B, [C, D, E]] = [[A, B, C], D, E] + [C, [A, B, D], E] + [C, D, [A, B, E]]. \tag{61}$$

It should be pointed out that the FI (61) is not an operator identity. It holds only in special circumstances. It is easy to verify that the FI (61) does not hold for the following sine 3-algebra with the general parameter  $B$

$$\begin{aligned}
[T_{\mathbf{m}}, T_{\mathbf{n}}, T_{\mathbf{k}}] &= 2\mathbf{i}[\sin B(\mathbf{m} \times \mathbf{n} - \mathbf{n} \times \mathbf{k} - \mathbf{k} \times \mathbf{m}) \\
&\quad + \sin B(\mathbf{k} \times \mathbf{m} - \mathbf{m} \times \mathbf{n} - \mathbf{n} \times \mathbf{k}) \\
&\quad + \sin B(\mathbf{n} \times \mathbf{k} - \mathbf{k} \times \mathbf{m} - \mathbf{m} \times \mathbf{n})] T_{\mathbf{m} + \mathbf{n} + \mathbf{k}}.
\end{aligned} \tag{62}$$

However it is found that when particularized to the  $B = \frac{\pi}{2}$  case, (62) becomes a Filippov algebra which satisfies the FI (61) [45].

## 5 The many-body system in the lowest Landau level

In the previous section, we have investigated the  $W_{1+\infty}$   $n$ -algebra of the infinite conserved operators for an electron moving on a plane in presence of a constant perpendicular magnetic field. Let us turn to the case of  $N$  electrons [26]. The Hamiltonian is given by

$$\bar{H} = \frac{1}{2m} \sum_{k=1}^N [\mathbf{p}_k - \frac{e}{c} \mathbf{A}(\mathbf{r}_k)]^2. \tag{63}$$

We may construct the harmonic oscillator operators  $a$  and  $a^\dagger$ ,

$$a_k = \frac{z_k}{2} + \hbar \bar{\partial}_k, \quad a_k^\dagger = \frac{\bar{z}_k}{2} - \hbar \partial_k, \tag{64}$$

obeying

$$[a_k, a_l^\dagger] = \hbar \delta_{kl}, \tag{65}$$

where  $z_k = x_k + iy_k$  is the complex coordinate for the location of the  $k$ th electron.

In the appropriately chosen system of units  $c = m = 1$ ,  $eB = 2$  and symmetric gauge  $\mathbf{A} = \frac{B}{2}(-y, x)$ , in terms of harmonic oscillator operators (64), the Hamiltonian (63) can be rewritten as

$$\bar{H} = \sum_{k=1}^N (a_k a_k^\dagger + a_k^\dagger a_k). \quad (66)$$

We may also express the angular momentum of  $N$  electrons as

$$\bar{J} = \hbar \sum_{k=1}^N (z_k \partial_k - \bar{z}_k \bar{\partial}_k) = \sum_{k=1}^N [b_k^\dagger b_k - a_k^\dagger a_k], \quad (67)$$

where

$$b_k = \frac{\bar{z}_k}{2} + \hbar \partial_k, \quad b_k^\dagger = \frac{z_k}{2} - \hbar \bar{\partial}_k, \quad (68)$$

obeying

$$[b_k, b_l^\dagger] = \hbar \delta_{kl}. \quad (69)$$

Let us take the operators [26]

$$\bar{W}_m^r = \sum_{i=1}^N (b_i^\dagger)^{m+r-1} (b_i)^{r-1}, \quad (70)$$

which are commuting with the Hamiltonian (63). For the triple operators  $(\bar{W}_m^r, \bar{J}, \bar{H})$ , we also have (54) and (55).

Straightforward calculation gives the following product of the conserved operators:

$$\begin{aligned} \bar{W}_{m_1}^{r_1} \bar{W}_{m_2}^{r_2} &= \sum_{\alpha=0}^{r_1-1} C_{r_1-1}^\alpha A_{m_2+r_2-1}^\alpha \hbar^\alpha \bar{W}_{m_1+m_2}^{r_1+r_2-1-\alpha} \\ &+ \sum_{\substack{i,j=1 \\ i \neq j}}^N (b_i^\dagger)^{m_1+r_1-1} (b_j^\dagger)^{m_2+r_2-1} (b_i)^{r_1-1} (b_j)^{r_2-1}. \end{aligned} \quad (71)$$

Note that when  $N > 1$ , the second term on the right hand side of (71) can not be expressed as the combinations of the conserved operators.

By calculating the commutation relations, it is found that the infinite conserved operators (70) also form the  $W_{1+\infty}$  algebra as for the case of single electron [26]. However the multibrackets for these operators are not the same as for single electron ones. Due to the second term on the right hand side of (71), these extra terms will appear in the case of the multibrackets although

they can be deleted from the commutation relations. Thus the infinite conserved operators (70) do not yield the closed  $n$ -algebra.

As an example, let us consider the following multibracket:

$$[\bar{W}_{m_1}^{n+1}, \bar{W}_{m_2}^{n+1}, \dots, \bar{W}_{m_{2nN}}^{n+1}] = ((2n)!)^{-N} \epsilon_{12 \dots (2nN)}^{i_1 i_2 \dots i_{2nN}} C_1 \prod_{j=0}^{N-1} V_{2n}(m_{i_{2nj+1}}, \dots, m_{i_{2n(j+1)}}) b_{j+1}^{\dagger m_{i_{2nj+1}} + \dots + m_{i_{2n(j+1)}} + n} b_{j+1}^n, \quad (72)$$

$$\text{where } C_1 = \prod_{i=1}^{N-1} (-1)^{n(2n+1)} \sum_{\substack{1 \leq j_{2n(i-1)+1} \\ < \dots < j_{2ni} \leq \\ 2nN - 2n(i-1)}} (-1)^{j_{2n(i-1)+1} + \dots + j_{2ni}}.$$

When  $N = 1$ , by taking the appropriate scaled generators, (72) becomes the sub- $2n$ -algebra (51). However, it is obvious that when  $N > 1$ , (72) does not form the closed generalized Lie algebra. In spite of this negative result, we observe that there exist the null sub- $n$ -algebras for the case of single electron. It is instructive to pursue the analysis of the case of  $N$  electrons. Let us continue to consider the case of  $(2nN + 1)$ -bracket.

$$\begin{aligned} [\bar{W}_{m_1}^{n+1}, \dots, \bar{W}_{m_{2nN+1}}^{n+1}] &= \left( \prod_{k=1}^N (l_k)! \right)^{-1} \epsilon_{12 \dots (2nN+1)}^{i_1 i_2 \dots i_{2nN+1}} \underbrace{[(b_1^\dagger)^{m_{i_1}+n} (b_1)^n, \dots, (b_1^\dagger)^{m_{i_{l_1}}+n} (b_1)^n], \dots,}_{l_1} \\ &\quad \underbrace{(b_j^\dagger)^{m_{i_{l_1}+\dots+l_{j-1}+1}+n} (b_j)^n, \dots, (b_j^\dagger)^{m_{i_{l_1}+\dots+l_{j-1}+l_j}+n} (b_j)^n, \dots,}_{l_j} \\ &\quad \underbrace{(b_N^\dagger)^{m_{i_{l_1}+\dots+l_{N-1}+1}+n} (b_N)^n, \dots, (b_N^\dagger)^{m_{i_{l_1}+\dots+l_{N-1}+l_N}+n} (b_N)^n}_{l_N} \\ &= \left( \prod_{k=1}^N (l_k)! \right)^{-1} \epsilon_{12 \dots (2nN+1)}^{i_1 i_2 \dots i_{2nN+1}} C \underbrace{[(b_1^\dagger)^{m_{i_1}+n} (b_1)^n, \dots, (b_1^\dagger)^{m_{i_{l_1}}+n} (b_1)^n]}_{l_1} \times \dots \\ &\quad \times \underbrace{[(b_j^\dagger)^{m_{i_{l_1}+\dots+l_{j-1}+1}+n} (b_j)^n, \dots, (b_j^\dagger)^{m_{i_{l_1}+\dots+l_{j-1}+l_j}+n} (b_j)^n]}_{l_j} \times \dots \\ &\quad \times \underbrace{[(b_N^\dagger)^{m_{i_{l_1}+\dots+l_{N-1}+1}+n} (b_N)^n, \dots, (b_N^\dagger)^{m_{i_{l_1}+\dots+l_{N-1}+l_N}+n} (b_N)^n]}_{l_N}, \quad (73) \end{aligned}$$

$$\text{where } C = \prod_{i=1}^{N-1} (-1)^{\frac{l_i(l_i+1)}{2}} \sum_{\substack{1 \leq j_{l_1+\dots+l_{i-1}+1} < \dots \\ < j_{l_1+\dots+l_{i-1}+l_i} \leq \\ 2nN+1-(l_1+\dots+l_{i-1})}} (-1)^{j_{l_1+\dots+l_{i-1}+1} + \dots + j_{l_1+\dots+l_{i-1}+l_i}}, \quad \sum_{i=1}^N l_i = 2nN + 1$$

and  $0 \leq l_i \leq 2nN + 1$ .

Since there exists  $l_i$  in (73) such that  $l_i \geq 2n + 1$ , by means of (14), we have

$$[\bar{W}_{m_1}^{n+1}, \bar{W}_{m_2}^{n+1}, \dots, \bar{W}_{m_{2nN+1}}^{n+1}] = 0. \quad (74)$$

Here we should like to draw attention to the fact that the symmetry (74) is not only determined by the superindex of the generators, but also the number of electrons  $N$ . It represents a remarkable property of the many-body system (63). In the limit  $N \rightarrow \infty$ , it can be observed that there are no closed  $n$ -ary algebras for the conserved operators, except for the  $W_{1+\infty}$  algebra.

Due to (74), it is easy to see that

$$[\bar{W}_{m_1}^{n+1}, \bar{W}_{m_2}^{n+1}, \dots, \bar{W}_{m_k}^{n+1}] = 0, \quad k \geq 2nN + 1, \quad (75)$$

and

$$[\bar{W}_{m_1}^{n+1}, \dots, \bar{W}_{m_k}^{n+1}, \bar{H}] = 0, \quad k \geq 2nN + 1. \quad (76)$$

For the case of single electron  $N = 1$ , (76) gives (57).

It is well-known that the filling factor  $\nu = 1$  ground state is given by the wave function

$$\psi_1(z_1, \dots, z_N) = \prod_{1 \leq i < j \leq N} (z_i - z_j) \exp\left(-\frac{1}{2\hbar} \sum_{i=1}^N |z_i|^2\right). \quad (77)$$

The action of generators  $W_m^r$  on the ground state wave function is given by

$$\begin{aligned} \bar{W}_m^r \psi_1 &= (r-1)! \sum_{1 \leq j_0 < j_1 < \dots < j_{r-1} \leq N} \begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ z_{j_0} & z_{j_1} & \dots & z_{j_{r-2}} & z_{j_{r-1}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_{j_0}^{r-2} & z_{j_1}^{r-2} & \dots & z_{j_{r-2}}^{r-2} & z_{j_{r-1}}^{r-2} \\ z_{j_0}^{r-1+m} & z_{j_1}^{r-1+m} & \dots & z_{j_{r-2}}^{r-1+m} & z_{j_{r-1}}^{r-1+m} \end{vmatrix} \psi_1 \\ &= (r-1)! \sum_{1 \leq j_0 < j_1 < \dots < j_{r-1} \leq N} S_{(m, 0, \dots, 0)}(z_{j_0}, z_{j_1}, \dots, z_{j_{r-2}}, z_{j_{r-1}}) \psi_1, \end{aligned} \quad (78)$$

where  $S_{(m, 0, \dots, 0)}$  is the Schur function corresponding to the Young diagram  $(m, \overbrace{0, \dots, 0}^{r-1})$ . Form (78), we have

$$\bar{W}_m^r \psi_1 = 0, \quad -r \leq m \leq -1, \quad r \geq 1. \quad (79)$$

Moreover the state  $\psi_1$  is the eigenstate for the zero modes

$$\bar{W}_0^r \psi_1 = (r-1)! C_N^r \psi_1. \quad (80)$$

From the action of (74) on the ground state wave function, i.e.,

$$[\bar{W}_{m_1}^{n+1}, \bar{W}_{m_2}^{n+1}, \dots, \bar{W}_{m_{2nN+1}}^{n+1}] \psi_1 = 0, \quad (81)$$

we obtain the following equality with respect to the Schur function:

$$\begin{aligned} & \epsilon_{12 \dots (2nN+1)}^{i_{2nN+1} \dots i_2 i_1} \sum_{k_2, \dots, k_{2nN+1}=1}^n [n! C_n^{k_2} \dots C_n^{k_{2nN+1}} D_{2nN+1} (D_{2nN} (\dots (D_2 \bar{S}_{m_{i_1}}) \dots)) \\ & + \sum_{s=1}^{nN} (n!)^{s+1} \sum_{\substack{3 \leq l_1 < \dots < l_s \leq 2nN+1 \\ l_{p+1} - l_p \geq 2 \\ 1 \leq p \leq s-1}} \frac{C_n^{k_2} \dots C_n^{k_{2nN+1}}}{C_n^{k_{l_1}} C_n^{k_{l_2}} \dots C_n^{k_{l_s}}} D_{2nN+1} (D_{2nN} (\dots D_{l_s+1} (\bar{S}_{m_{i_{l_s}}} D_{l_s-1} (\dots \\ & D_{l_{s-1}} (\bar{S}_{m_{i_{l_{s-1}}}) D_{l_{s-1}-1} (\dots D_{l_1+1} (\bar{S}_{m_{i_{l_1}}} D_{l_1-1} (\dots (D_2 \bar{S}_{m_{i_1}}) \dots)) \dots)) \dots)))] = 0, \quad (82) \end{aligned}$$

where

$$\bar{S}_m = \sum_{1 \leq j_0 < j_1 < \dots < j_n \leq N} S_{(m, 0, \dots, 0)}(z_{j_0}, z_{j_1}, \dots, z_{j_{n-1}}, z_{j_n}), \quad (83)$$

and

$$D_p = \sum_{j_p=1}^N f_{m_{i_p}, k_p}^n(z_{j_p}) \partial_{z_{j_p}}^{k_p}, \quad 2 \leq p \leq 2nN+1, \quad (84)$$

in which

$$f_{m_{i_p}, k_p}^n(z_{j_p}) = z_{j_p}^{m_{i_p}+n} \sum_{l_1, \dots, l_{n-k_p}=1}^{N'} \frac{1}{(z_{j_p} - z_{l_1})(z_{j_p} - z_{l_2}) \dots (z_{j_p} - z_{l_{n-k_p}})}, \quad k_p \geq 1, \quad (85)$$

the prime on the sum is used to indicate that any two summation indices do not coincide. The relation between the symmetry (74) and the equality (82) provides additional insight into the property of the ground state wave function (77).

## 6 Summary

We have presented the  $W_{1+\infty}$   $n$ -algebra which satisfies the GJI and GBI for the  $n$  even and odd cases, respectively. For the generators  $W_m^r$  (1) of  $W_{1+\infty}$  ( $n$ -)algebra, it is well-known that the generators  $W_m^2$  yield the Virasoro-Witt algebra, and the remaining generators do not yield any nontrivial sub-algebra. Our key finding is that the generators  $W_m^{n+1}$  with any fixed superindex  $n+1 > 2$  yield the null sub- $k$ -algebras for  $k \geq 2n+1$  and the nontrivial sub- $2n$ -algebra in which the structure constants are determined by the Vandermonde determinant. The sub- $2n$ -algebra is the so-called generalized Lie algebra which satisfies the GJI. Due to the remarkable property of



the structure constants, we also derived the centrally extended  $W_{1+\infty}$  sub- $2n$ -algebra. Not as the case of the sub- $2n$ -algebra, the form of  $W_{1+\infty}$   $2n$ -algebra appears to become more complicated. Therefore it still remains an open question whether there exist the central extension terms for the general  $W_{1+\infty}$   $2n$ -algebra.

It is well known that there exist infinite conserved operators in Landau problem, which yield the  $W_{1+\infty}$  algebra. Based on the quantum multibrackets, we found that these infinite conserved operators also yield the  $W_{1+\infty}$   $n$ -algebra. A remarkable property is that in the classical limit, these infinite-dimensional symmetries do not only reduce to the well-known classical  $w_\infty$  algebra but also give the so-called  $w_\infty$  3-algebra. We also noted that the  $W_{1+\infty}$  sub- $2n$ -algebra contains only a unique term in  $\hbar^{n(2n-1)}$ , which represents the higher-order quantum effect. It explores the intrinsic symmetry in higher order of  $\hbar$ . Moreover we investigated the case of the many-body problem in the lowest Landau level, where the corresponding infinite conserved operators also form the  $W_{1+\infty}$  algebra as for the case of single electron. However we noted that the quantum multibrackets for these operators do not yield the nontrivial  $n$ -algebras. But for the infinite conserved operators with any fixed superindex, we found that there exist the null multibrackets which are not only determined by the superindex, but also the number of electrons. It turns out that there are more symmetries than have been previously recognized for the infinite conserved operators of the many-body system (63). From the action of the null quantum multibracket on the  $\nu = 1$  ground state wave function, we derived the equality with respect to the Schur function. Our analysis provides additional insight into the  $W_{1+\infty}$   $n$ -algebra. Due to its remarkable properties, more applications in physics should be of interest.

## Acknowledgements

This work is partially supported by the NSFC projects (11375119, 11475116 and 11605096).

## References

- [1] J. Bagger and N. Lambert, Modeling multiple M2's, Phys. Rev. D **75** (2007) 045020 [hep-th/0611108]; Gauge symmetry and supersymmetry of multiple M2-branes, Phys. Rev. D **77** (2008) 065008 [arXiv:0711.0955]; Comments on multiple M2-branes, JHEP **02** (2008) 105 [arXiv:0712.3738].
- [2] A. Gustavsson, Algebraic structures on parallel M2-branes, Nucl. Phys. B **811** (2009) 66 [arXiv:0709.1260].

- [3] B. Estienne, N. Regnault and B.A. Bernevig,  $D$ -algebra structure of topological insulators, Phys. Rev. B **86** (2012) 241104(R) [arXiv:1202.5543].
- [4] T. Neupert, L. Santos, S. Ryu, C. Chamon and C. Mudry, Noncommutative geometry for three-dimensional topological insulators, Phys. Rev. B **86** (2012) 035125.
- [5] K. Hasebe, Chiral topological insulator on Nambu 3-algebraic geometry, Nucl. Phys. B **886** (2014) 681.
- [6] K. Hasebe, Higher dimensional quantum Hall effect as  $A$ -class topological insulator, Nucl. Phys. B **886** (2014) 952 [arXiv:1403.5066].
- [7] T.L. Curtright, D.B. Fairlie and C.K. Zachos, Ternary Virasoro-Witt algebra, Phys. Lett. B **666** (2008) 386 [arXiv:0806.3515].
- [8] T. Curtright, D. Fairlie, X. Jin, L. Mezincescu and C. Zachos, Classical and quantal ternary algebras, Phys. Lett. B **675** (2009) 387 [arXiv:0903.4889].
- [9] S. Chakraborty, A. Kumar and S. Jain,  $w_\infty$  3-algebra, JHEP **09** (2008) 091 [arXiv:0807.0284].
- [10] M.R. Chen, K. Wu and W.Z. Zhao, Super  $w_\infty$  3-algebra, JHEP **09** (2011) 090 [arXiv:1107.3295].
- [11] M. Axenides and E. Floratos, Nambu-Lie 3-algebras on fuzzy 3-manifolds, JHEP **02** (2009) 039 [arXiv:0809.3493].
- [12] C.N. Pope, L.J. Romans and X. Shen, The complete structure of  $W_\infty$ , Phys. Lett. B **236** (1990) 173.
- [13] C.N. Pope, L.J. Romans and X. Shen,  $W_\infty$  and the Racah-Wigner algebra, Nucl. Phys. B **339** (1990) 191.
- [14] L.D. Landau, Diamagnetismus der Metalle, Z. Phys. **64** (1930) 629.
- [15] F.D.M. Haldane, Fractional quantization of the Hall effect: A hierarchy of incompressible quantum fluid states, Phys. Rev. Lett. **51** (1983) 605.
- [16] K. Hasebe and Y. Kimura, Fuzzy supersphere and supermonopole, Nucl. Phys. B **709** (2005) 94 [arXiv:0409230].
- [17] K. Hasebe, Quantum Hall liquid on a noncommutative superplane, Phys. Rev. D **72** (2005) 105017 [arXiv:0503162].

- [18] E. Ivanov, L. Mezincescu, P.K. Townsend, Planar super-Landau models, JHEP **01** (2006) 143 [arXiv:0510019].
- [19] T. Curtright, E. Ivanov, L. Mezincescu and P.K. Townsend, Planar super-Landau models revisited, JHEP **04** (2007) 020 [arXiv:0612300].
- [20] A. Beylin, T. Curtright, E. Ivanov, L. Mezincescu and P.K. Townsend, Unitary spherical super-Landau models, JHEP **10** (2008) 069 [arXiv:0806.4716].
- [21] E. Ivanov, Supersymmetrizing Landau models, Theor. Math. Phys. **154** (2008) 349 [arXiv:0705.2249].
- [22] A. Beylin, T. Curtright, E. Ivanov and L. Mezincescu, Generalized  $N = 2$  super Landau models, JHEP **04** (2010) 091 [arXiv:1003.0218].
- [23] V. Bychkov and E. Ivanov,  $N = 4$  supersymmetric Landau models, Nucl. Phys. B **863** (2012) 33.
- [24] R.E. Prange and S.M. Girvin, The quantum Hall effect, Springer, Berlin, 1990.
- [25] M. Stone, Quantum Hall effect, World Scientific, Singapore, 1992.
- [26] A. Cappelli, C.A. Trugenberger and G.R. Zemba, Infinite symmetry in the quantum Hall effect, Nucl. Phys. B **396** (1993) 465.
- [27] I.I. Kogan, Area-preserving diffeomorphisms,  $W_\infty$  and  $u_q[sl(2)]$  in Chern-Simons theory and the quantum Hall system, Int. J. Mod. Phys. A **9** (1994) 3887.
- [28] A. Capelli, C. Trugenberger and G. Zemba, Classification of Quantum Hall Universality Classes by  $W_{1+\infty}$  Symmetry, Phys. Rev. Lett. **72** (1994) 1902; Large  $N$  limit in the quantum Hall effect, Phys. Lett. B **306** (1993) 100.
- [29] S. Iso, D. Karabali and B. Sakita, Fermions in the lowest Landau level. Bosonization,  $W_\infty$  algebra, droplets, chiral bosons, Phys. Lett. B **296** (1992) 143.
- [30] H. Azuma,  $W_\infty$  algebra in the integer quantum Hall effects, Prog. Theor. Phys. **92** (1994) 293.
- [31] D. Karabali, Algebraic aspects of the fractional quantum Hall effect, Nucl. Phys. B **419** (1994) 437.
- [32] J.A. de Azcárraga and J.M. Izquierdo,  $n$ -ary algebras: a review with applications, J. Phys. A: Math. Theor. **43** (2010) 293001.

- [33] T. Curtright, X. Jin and L. Mezincescu, Multi-operator brackets acting thrice, J. Phys. A: Math. Theor. **42** (2009) 462001.
- [34] C. Devchand, D. Fairlie, J. Nuyts and G. Weingart, Ternutator identities, J. Phys. A: Math. Theor. **42** (2009) 475209 [arXiv:0908.1738].
- [35] C.N. Pope, L.J. Romans and X. Shen, A new higher-spin algebra and the lone-star product, Phys. Lett. B **242** (1990) 401; Ideals of Kac-Moody algebras and realisations of  $W_\infty$ , Phys. Lett. B **245** (1990) 72.
- [36] F. Yu and Y.S. Wu, Hamiltonian structure, (anti-)self-adjoint flows in the KP hierarchy and the  $W_{1+\infty}$  and  $W_\infty$  algebras, Phys. Lett. B **263** (1991) 220.
- [37] K. Yamagishi, A hamiltonian structure of KP hierarchy,  $W_{1+\infty}$  algebra, and self-dual gravity, Phys. Lett. B **259** (1991) 436.
- [38] F. Martínez-Morás and J. Mas, Centrally extended  $W_{1+\infty}$  and the KP hierarchy, Phys. Lett. B **344** (1995) 127.
- [39] M.R. Bremner, Identities for the ternary commutator, J. Algebra **206** (1998) 615.
- [40] M.R. Bremner and L.A. Peresi, Ternary analogues of Lie and Malcev algebras, Linear Algebra Appl. **414** (2006) 1.
- [41] V.T. Filippov, n-Lie algebras, Sib. Math. J. **26** (1985) 879.
- [42] T. Curtright and C. Zachos, Classical and quantum Nambu mechanics, Phys. Rev. D **68** (2003) 085001.
- [43] D.B. Fairlie and C.K. Zachos, Infinite-dimensional algebras, sine brackets, and  $SU(\infty)$ , Phys. Lett. B **224** (1989) 101.
- [44] D.B. Fairlie, P. Fletcher and C.K. Zachos, Trigonometric structure constants for new infinite-dimensional algebras, Phys. Lett. B **218** (1989) 203.
- [45] L. Ding, X.Y. Jia, K. Wu, Z.W. Yan, and W.Z. Zhao, On  $q$ -deformed infinite-dimensional  $n$ -algebra, Nucl. Phys. B **904** (2016) 18 [arXiv:1404.0464].